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A BOUNDARY ELEMENT METHOD FOR A TWO-DIMENSIONAL INTERFACE PROBLEM IN ELECTROMAGNETICS

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ABSTRACT

This paper presents Galerkin approximations for solutions of two dimensional interface problems by solving corresponding boundary integral equations. These are obtained by simple layer potential operators only. Due to the strong ellipticity of the integral equations the Galerkin procedure converges with optimal order. Smoothness of the given data implies high convergence rates for the layers.

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1. Introduction

Recently for the problem of the scattering of time harmonic electromagnetic fields by metallic obstacles, the eddy current problem, Hariharan and MacCamy [3], obtained two-dimensional results using boundary integral equations in the context of classical function spaces. They studied the interface problem (P_β) for the potential ϕ of the electric field induced by a thin wire carrying a pericdic current. They obtained a system of boundary integral equations whose solution determines the solution of (P_β). The goals of this paper are (i) to solve the same system with less regularity assumptions in appropriate Sobolev spaces and (ii) to obtain theoretical error estimates for the Galerkin method when the system of boundary integral equations is solved approximately with a regular finite elements method. A corresponding numerical Galerkin-collocation method based on our theory will be reported elsewhere.

We shall begin with a brief description of the physical problem. Let Ω be a simply connected bounded region in $\mathbb{R}^2$ and $\Omega^+ = \mathbb{R}^2 \setminus \bar{\Omega}$. Here, Ω^+ is to represent air and Ω cross section of a metallic cylinder in the $x-y$ plane (see Figure 1). We suppose there are incident electric and magnetic fields $\underset{\sim}{E}^0$, $\underset{\sim}{H}^0$, and that all fields $\underset{\sim}{E}$, $\underset{\sim}{H}$ are transverse magnetic and time harmonic with a single angular frequency ω. This means that, with a proper choice of x, y, z axes, $\underset{\sim}{E} = E\hat{k}$, $\underset{\sim}{H} = H^1\hat{i} + H^2\hat{j}$ where E, H^1, H^2 are functions of x and y only. Our underlying assumptions are that the conductivity of air is zero and displacement current can be neglected in metal. $\underset{\sim}{E}$ and $\underset{\sim}{H}$ satisfy Maxwell's equations in Ω^+ and Ω and are subject to the interface condition that the tangential components of $\underset{\sim}{E}$ and $\underset{\sim}{H}$ are continuous across Γ the boundary of Ω. Then, with appropriate scaling, we have the following problem:

$$(M)\quad \begin{cases} \operatorname{curl} \mathbf{E} = \mathbf{H} \ , & \operatorname{curl} \mathbf{H} = i\,\alpha^2\, \mathbf{E} \quad \text{in } \Omega \\ \operatorname{curl} \mathbf{E} = \mathbf{H} \ , & \operatorname{curl} \mathbf{H} = \beta^2\, \mathbf{E} \quad \text{in } \Omega^+ \\ \mathbf{n} \times \mathbf{E},\ \mathbf{n} \times \mathbf{H} \ \text{continuous across } \Gamma. & \end{cases}$$

$\mathbf{n}$ denotes the exterior normal to Ω and α, β are non-dimensional constants. We suppose there is a wire parallel to the z-axis through $\mathbf{x}_0 \in \Omega^+$ carrying a periodic current $I(t) = \mathrm{Re}\{I_0 e^{-i\omega t}\}$, $I_0 \in \mathbb{C}$. Then the incident field $\mathbf{E}^0$, $\mathbf{H}^0$ due to the wire will have the form

$$\mathbf{E}^0(x,y,z,t) = \mathrm{Re}\{I_0\, \phi_\beta(x,y)\, \hat{k}\, e^{-i\omega t}\}$$

$$\mathbf{H}^0(\mathbf{x},y,z,t) = \mathrm{Re}\{I_0 (\frac{\partial\phi_\beta}{\partial y} \hat{i} - \frac{\partial\phi_\beta}{\partial x} \hat{j}) e^{-i\omega t}\}$$

with the Hankel function of first kind and of order zero

$$\phi_\beta(x) = -\frac{i}{4} H_0^{(1)}(\beta|\mathbf{x} - \mathbf{x}_0|), \quad \mathbf{x} = (x,y). \tag{1.1}$$

The fields $\mathbf{E}^0$, $\mathbf{H}^0$ satisfy Maxwell's equations (M) in $\Omega^+\{x_0\}$ (air). If we seek the total fields $\mathbf{E}$, $\mathbf{H}$ in the same form with ϕ_β replaced by the solution ϕ of the interface problem (P_β), then $\mathbf{E}$, $\mathbf{H}$ satisfy the remaining Maxwell's equations and are automatically divergence free and $\mathbf{E} - \mathbf{E}^0$, $\mathbf{H} - \mathbf{H}^0$ satisfy a radiation condition. Thus, altogether the two-dimensional eddy current problem for a conducting cylinder is reduced to the interface problem:

Find $\phi \in C^2(\Omega) \cup C^2(\Omega^+ \setminus \{x_0\}) \cup C^0(\mathbb{R}^2)$ such that,

$$(P_\beta)\qquad \begin{aligned} \Delta\phi + i\alpha^2\phi &= 0 \quad \text{in } \Omega \ , \\ \Delta\phi + \beta^2\phi &= 0 \quad \text{in } \Omega^+ \ , \end{aligned}$$

where $\phi, \frac{\partial\phi}{\partial n} \in C^0(\Gamma)$ and $\phi - \phi_\beta \in C^2(\Omega^+) \cup C^0(\Gamma)$ satisfying Sommerfeld's radiation condition. Here ϕ_β as in (1.1) represents the potential induced by the wire (located at a point $x_0 \in \Omega^+$).
Note that in (P_β) the parameters α, β are real numbers, β being sufficiently small, namely $\alpha^2 = 0(10)$, $\beta^2 = 0(10^{-21})$ (see [2]).

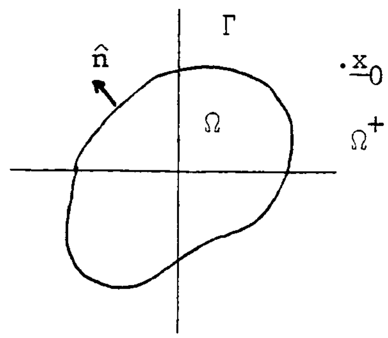

Figure 1.

Therefore, instead of (P_β) we consider:

Find $\psi \in C^2(\Omega) \cup C^2(\Omega^+ \setminus \{x_0\}) \cup C^0(\Gamma)$ such that,

$$(P_0) \qquad \begin{aligned} \Delta\psi + i\alpha^2\psi &= 0 \quad \text{in } \Omega , \\ \Delta\psi &= 0 \quad \text{in } \Omega^+, \end{aligned}$$

where $\psi, \frac{\partial\psi}{\partial n} \in C^0(\Gamma)$ and $\psi - \psi_0 \in C^2(\Omega^+) \cup C^0(\Gamma)$, ψ being bounded at infinity.

It was shown in [2] that with

$$\psi_0(\underset{\sim}{x}) = \frac{1}{2\pi} \log|\underset{\sim}{x} - \underset{\sim}{x}_0| ,$$

the solution ϕ of (P_β) converges pointwise to the solution ψ of (P_0) as $\beta \to 0$ at any fixed $\underset{\sim}{x} \in \mathbb{R}^2 \setminus \{x_0\}$. Furthermore by [2], [3] problem (P_0) can be transformed into a coupled system of integral equations on the boundary Γ by the use of simple layer potentials, only. Thus, the solution ψ of (P_0) was constructed by continuous densities f and g on Γ (see [2]), namely:

$$\psi(\underset{\sim}{x}) = \begin{cases} -\frac{i}{4}\int_\Gamma f(\underset{\sim}{y})H_0^{(1)}(\sqrt{i}\,\alpha|\underset{\sim}{x}-\underset{\sim}{y}|)ds_{\underset{\sim}{y}}, & \underset{\sim}{x} \in \Omega \quad , \\ \frac{1}{2\pi}\int_\Gamma g(\underset{\sim}{y})\log|\underset{\sim}{x}-\underset{\sim}{y}|ds_{\underset{\sim}{y}} + \psi_0(\underset{\sim}{x}) + C, & \underset{\sim}{x} \in \Omega^+ \quad , \end{cases} \tag{1.2}$$

where ψ_0 is given as above and $C \in \mathbb{C}$, arbitrary. The regularity assumptions that $\psi, \frac{\partial\psi}{\partial n} \in C^0(\Gamma)$ and $\psi(\infty)$ being bounded yield a coupled system of integral equations on Γ.

$$\text{(I)} \qquad \begin{cases} V_\alpha(f) - \psi_0 = V_0(g) + C \; , \\ -1 = \int_\Gamma g\, ds \; , \\ -\frac{1}{2}f + N_\alpha(f) = \frac{1}{2}g + N_0(g) + \frac{\partial\psi_0}{\partial n} \quad . \end{cases}$$

In (I) for $\underset{\sim}{x} \in \Gamma$ and $f \in C^0(\Gamma)$ we have

$$\begin{cases} V_\alpha(f)\underset{\sim}{x} := -\frac{i}{4}\int_\Gamma f(\underset{\sim}{y})H_0^{(1)}(\sqrt{i}\,\alpha|\underset{\sim}{x}-\underset{\sim}{y}|)ds_{\underset{\sim}{y}} \; , \\ V_0(f)\underset{\sim}{x} := \frac{1}{2\pi}\int_\Gamma f(\underset{\sim}{y})\log|\underset{\sim}{x}-\underset{\sim}{y}|ds_{\underset{\sim}{y}} \; , \\ N_\alpha(f)\underset{\sim}{x} := \frac{\partial}{\partial n_{\underset{\sim}{x}}} V_\alpha f(\underset{\sim}{x}) = -\frac{i}{4}\int_\Gamma f(\underset{\sim}{y})\frac{\partial}{\partial n_{\underset{\sim}{x}}} H_0^{(1)}(\sqrt{i}\,\alpha|\underset{\sim}{x}-\underset{\sim}{y}|)ds_{\underset{\sim}{y}} \; , \\ N_0(f)\underset{\sim}{x} := \frac{\partial}{\partial n_{\underset{\sim}{x}}} V_0 f(\underset{\sim}{x}) = \frac{1}{2\pi}\int_\Gamma f(\underset{\sim}{y})\frac{\partial}{\partial n_{\underset{\sim}{x}}}\log|\underset{\sim}{x}-\underset{\sim}{y}|ds_{\underset{\sim}{y}} \quad . \end{cases} \tag{1.3}$$

In [2] it is proved that the more general system

$$
(\mathrm{I}^*) \qquad \begin{cases} V_\alpha(f) - V_0(g) - c = h_1 \;, \\ \int_\Gamma g\,ds = a \;, \\ (-\frac{1}{2} + N_\alpha)f - (\frac{1}{2} + N_0)g = h_2 \;, \end{cases}
$$

is uniquely solvable for given $(h_1, h_2, a) \in C^1(\Gamma) \times C^0(\Gamma) \times \mathbb{R}$ with solution $(f,g,c) \in C^0(\Gamma) \times C^0(\Gamma) \times \mathbb{C}$ yielding the existence and uniqueness of the solution of (I). The proof was performed by converting (I*) into an uncoupled system of Fredholm integral equations of the second kind for the unknown densities $f,g \in C^0(\Gamma)$.

In the following we solve (I) under less regularity assumptions in appropriate Sobolev spaces. Also we obtain the proof of theoretical error estimates for the Galerkin method when the system (I) of boundary integrals is solved approximately with regular finite elements on Γ. Considering (I) in Sobolev spaces it turns out that the system (I) is strongly elliptic in the sense of [17], i.e., it satisfies a Gårding's inequality. Therefore, the Galerkin approximation converges for mesh size $h \to 0$ to the exact solution of (I) with optimal order due to [14]. Furthermore, we want to mention that exterior interface problems in this context are solved with variational methods employing nonlocal boundary conditions in [9], [10].

2. Unique Solvability of the Integral Equations in Sobolev Spaces

First we consider the system (I*) for data given in Sobolev spaces on Γ. Then the operators involved have the following mapping properties.

<u>Lemma 2.1</u>: Let $\Gamma \in C^{\ell}$, $\ell \geq |s| + 2$. Then for any $s \in \mathbb{R}$ the operators given in (1.3) are bounded mappings as follows:

$$V_\alpha : H^s(\Gamma) \to H^{s+1}(\Gamma) \ , \tag{2.1}$$

$$V_0 : H^s(\Gamma) \to H^{s+1}(\Gamma) \ , \tag{2.2}$$

$$N_\alpha : H^s(\Gamma) \to H^{s+(\ell-|s|-2)}(\Gamma) \ , \tag{2.3}$$

$$N_0 : H^s(\Gamma) \to H^{s+(\ell-|s|-2)}(\Gamma) \ . \tag{2.4}$$

<u>Proof</u>: For (2.1) we use the expansion of the kernel

$$-\frac{i}{4} H_0^{(1)}(z) = \frac{1}{2\pi} \log z + \gamma + O(z^2 \log z), \qquad \gamma \in \mathbb{C}. \tag{2.5}$$

Thus with a smoothing operator R we can write

$$V_\alpha = V_0 + R \ ,$$

and application of the Fourier transform F shows that V_α and V_0 have the same principal symbol, namely

$$\sigma(V_\alpha) = \sigma(V_0) = -\tfrac{1}{2} \frac{1}{|\xi|} = F_{z \to \xi}\left(\frac{1}{2\pi} \log z\right) .$$

Hence, V_α and V_0 are pseudo-differential operators of order -1 yielding the proposed properties (2.1), (2.2) (see [17], [7]).

For (2.4) we note that N_0 is just the operator adjoint to the double layer potential and therefore it has a kernel $k(s,t) \in C^{\ell-2}(\Gamma \times \Gamma)$ implying the desired smoothness (2.4) (see [16, Lemma A.1, p. 309]).

For (2.3) we use that

$$\tilde{\tilde{R}}(f)\underset{\sim}{x}: = \int_{\Gamma} f(\underset{\sim}{y}) \frac{\partial}{\partial n_{\underset{\sim}{x}}}\{|\underset{\sim}{x}-\underset{\sim}{y}|^2 \log|\underset{\sim}{x}-\underset{\sim}{y}|\} ds_{\underset{\sim}{y}},$$

is a pseudo-differential operator of order-3 ([15]). On the other hand the remaining term in (2.5) gives the same contribution as the adjoint of the double layer potential. Thus (2.3) follows from (2.4).

For the solvability of (I*) the existence of the inverse of an auxiliary problem is crucial which comes from the exterior Dirichlet problem. Here we remark that the result of the following lemma is also proved in [5] and for Hölder spaces in [6].

Lemma 2.2: For any given $(h,a) \in H^s(\Gamma) \times \mathbb{R}$, $s \geq 1$, there exists exactly one solution $(g,c) \in H^{s-1}(\Gamma) \times \mathbb{C}$ of

$$V_0(g) + c = h, \qquad \int_{\Gamma} g\, ds = a. \tag{2.6}$$

Proof: For $(h,a) \in C^{1+\alpha}(\Gamma) \times \mathbb{R}$, $\alpha > 0$, it is shown in [6] that (2.6) has a unique solution in $C^{\alpha}(\Gamma) \times \mathbb{C}$, provided the condition that the mapping radius of Γ is not equal to one. It is shown in [2] that for $(h,a) \in C^1(\Gamma) \times \mathbb{R}$ (2.6) has a unique solution in $C^0(\Gamma) \times \mathbb{C}$ without this condition. We first give this proof in Hölder spaces. This is done in two steps.

First, there exists exactly one solution $(f_0, \gamma_0) \in C^{\infty}(\Gamma) \times \mathbb{R}$ satisfying

$$V_0(f_0) = \gamma_0, \qquad \int_{\Gamma} f_0\, ds = 1. \tag{2.7}$$

Furthermore for any $h \in C^{1+\alpha}(\Gamma)$ there exists exactly one solution $f_p \in C^{\alpha}(\Gamma)$ of

$$V_0(f_p) = h + \Gamma_1(h), \tag{2.8}$$

where the functional $\Gamma_1(h)$ is given by

$$\Gamma_1(h) = \frac{1}{L}\int_\Gamma V_0(f_p)(\sigma)d\sigma - \int_\Gamma h(\sigma)d\sigma, \quad L: = \text{length of } \Gamma \tag{2.9}$$

and σ is the arc length.

The uniqueness and existence of $f_p \in C^\alpha(\Gamma)$ was shown by differentiating (2.8) with respect to the arc length. Then (2.8) becomes

$$Hf_p + Rf_p = \frac{dh}{ds}, \qquad Hf_p: = \text{p.v.}\ \frac{1}{2\pi}\int_\Gamma \frac{f_p(\underset{\sim}{y})}{s_{\underset{\sim}{x}}-s_{\underset{\sim}{y}}}\, ds_{\underset{\sim}{y}}, \tag{2.10}$$

which is a singular integral equation with the Hilbert kernel plus a smooth remainder.

Upon applying the Hilbert transform, equation (2.10) becomes a Fredholm integral equation of second kind for f_p, since for any $f_p \in L^2(\Gamma)$

$$H(Hf_p) = -f_p \qquad \text{(see [12]).} \tag{2.11}$$

Thus we have

$$(I+C)f_p = H\frac{dh}{ds},$$

with a compact operator C on $H^{s-1}(\Gamma)$. Now, using the results of Muskhelishvili [11, p. 118 ff.] the solution of (2.10) can be represented as

$$f_p(\underset{\sim}{x}) = \text{p.v.}\int_\Gamma \left(\frac{1}{s_{\underset{\sim}{x}}-s_{\underset{\sim}{y}}} + r(\underset{\sim}{x},\underset{\sim}{y})\right)\frac{dh}{ds}(\underset{\sim}{y})\,ds_{\underset{\sim}{y}} = :M(h), \tag{2.12}$$

with a smooth kernel $r(\cdot,\cdot)$ depending only on the regularity of Γ.

Now the form (2.9) of $\Gamma_1(h)$ is easily obtained by inserting (2.12) into (2.8) and integrating over Γ.

For $h \in H^s(\Gamma)$ obviously $\frac{dh}{ds} \in H^{s-1}(\Gamma)$. The operator H is a singular integral operator of Cauchy type and therefore a pseudo-differential operator of order zero [8]. Therefore by density, the solution f_p of (2.12) belongs to $H^{s-1}(\Gamma)$, $s \geq 1$.

Finally with the solutions f_0, f_p of (2.7) and (2.8), respectively, the solution (g,c) of (2.6) is given by

$$g = f_p + \lambda f_0 \,, \tag{2.13}$$

with

$$\lambda = a - \int_\Gamma f_p ds, \tag{2.14}$$

and

$$c = -(\Gamma_1(h) + \lambda\gamma_0) \,. \tag{2.15}$$

Now we are in the position to formulate the main result of this section:

<u>Theorem 2.3</u>: For $s \in \mathbb{R}$, $s \geq 1$, and for any given $(h_1,h_2,a) \in H^s(\Gamma) \times H^{s-1}(\Gamma) \times \mathbb{R}$ the system (I*) has a unique solution $(f,g,c) \in H^{s-1}(\Gamma) \times H^{s-1}(\Gamma) \times \mathbb{C}$.

Proof: In (I*) let us set

$$h: = V_\alpha(f) - h_1, \tag{2.16}$$

for any $f \in H^{s-1}(\Gamma)$ and for any $h_1 \in H^s(\Gamma)$. Then by Lemma 2.2 there exists exactly one solution (g,c) of (2.6) which coincides with the first two equations in (I*). Due to the form (2.13) $g \in H^{s-1}(\Gamma)$ is given explicitly by

$$g = A(h): = M(h) + \lambda(h)f_0. \tag{2.17}$$

Therefore with (2.16) the equation (2.17) reads

$$g = \mathcal{D}(f) - A(h_1), \tag{2.18}$$

where

$$\mathcal{D}(f) : = A(V_\alpha(f)) = M(V_\alpha(f)) + \lambda(V_\alpha(f) - h_1)f_0. \tag{2.19}$$

Thus, after all, g and c given by (2.13) and (2.15), respectively, give the solution of the first two equations of (I*) in terms of f, h_1 and a. Now we can uncouple the system (I*) by inserting the explicit form (2.18) of g into the third equation of (I*), and we obtain

$$(2.20)\quad -\tfrac{1}{2}f + N_\alpha(f) = \tfrac{1}{2}(\mathcal{D}(f) - A(h_1)) + N_0(\mathcal{D}f) - N_0(A(h_1)) + h_2.$$

Due to Lemma 2.1 and the Rellich embedding theorem the operators N_α and $N_0\mathcal{D}$ are compact in $H^{s-1}(\Gamma)$ since $\mathcal{D}$ - defined by (2.18) and (2.12) - is bounded in $H^{s-1}(\Gamma)$.

Unfortunately, in order for $\mathcal{D}(f)$ to be given by (2.19) we have to insert the derivative of the Hankel function (2.5) into (2.12). According to (2.11) we obtain

$$(2.21)\qquad \mathcal{D}(f) = f + R^+(f) + W(f),$$

where R^+ is a smooth operator. With $W(f)$ we denote terms corresponding to the asymptotic expansion of $H_0^{(1)}(|\underset{\sim}{x} - \underset{\sim}{y}|)$ for small $|\underset{\sim}{x} - \underset{\sim}{y}|$. It turns out in [2 , p. 63, ff] that the kernel $k(s,t)$ of W behaves as

$$\text{p.v.}\int_\Gamma \frac{|\underset{\sim}{x}(s) - \underset{\sim}{x}(\sigma)|}{s - t}\left(\frac{\partial}{\partial s}\,|\underset{\sim}{x}(s) - \underset{\sim}{x}(\sigma)|\right)\log|\underset{\sim}{x}(s) - \underset{\sim}{x}(\sigma)|\,ds,$$

and therefore, it behaves as

$$k(s,t) = \text{p.v.}\int_\Gamma \frac{\log|\underset{\sim}{x}(s) - \underset{\sim}{x}(\sigma)|}{s - t}\,ds,$$

and thus

$$W(f)(\underset{\sim}{x}(s)) = \int_\Gamma k(s,t)f(t)dt,$$

which is the kernel of HV_0, this composition is a pseudo-differential operator of order-1 because H has the symbol $\sigma(H) = \operatorname{sgn}\xi$ and $\sigma(V_0) = \frac{-1}{2|\xi|}$; hence $HV_0 f \in H^s(\Gamma)$ for $f \in H^{s-1}(\Gamma)$.

So finally, with (2.21) the equation (2.20) turns out to be a Fredholm equation of the second kind for the unknown density $f \in H^{s-1}(\Gamma)$, namely:

$$(I + C_1)f = S_1(h_1, h_2) \tag{2.22}$$

where C_1 is a compact operator in $H^{s-1}(\Gamma)$. For given $h_1 \in H^s(\Gamma)$ and $h_2 \in H^{s-1}(\Gamma)$ obviously $S_1(h_1, h_2)$ belongs to $H^{s-1}(\Gamma)$; furthermore $S_1(0,0) = 0$, see [2].

Using potential theory it is shown in [2,p. 66,ff] that $I + C_1$ is injective due to the uniqueness of the original interface problem (P_0). Thus $(I + C_1)^{-1}$ exists and is a bounded operator in $H^{s-1}(\Gamma)$.

Therefore inserting now the explicitly given solution $f \in H^{s-1}(\Gamma)$ of (2.22) into the third equation of (I*) we obtain immediately

$$(I + 2N_0)g = S_2(h_1, h_2), \tag{2.23}$$

which again is a Fredholm equation of the second kind for $g \in H^{s-1}(\Gamma)$ due to (2.4). Again as above, potential theory shows that (2.23) is uniquely solvable.

Remark 2.4. Note that we have shown in the proof above that there holds the a-priori estimates with arbitrary constants C_1, $C_2 > 0$

$$\|f\|_{H^{s-1}(\Gamma)} \leq C_1 \left(\|h_1\|_{H^s(\Gamma)} + \|h_2\|_{H^{s-1}(\Gamma)} \right); \tag{2.24}$$

$$\|g\|_{H^{s-1}(\Gamma)} \leq C_2 \left(\|h_1\|_{H^s(\Gamma)} + \|h_2\|_{H^{s-1}(\Gamma)} \right). \tag{2.25}$$

Also, by the Fredholm property of the equations (2.22), (2.23) there holds a Gårding's inequality which is crucial for the convergence of the Galerkin approximation [14].

Lemma 2.5: For all $(f,g) \in H^{s-1}(\Gamma) \times H^{s-1}(\Gamma)$, $s \geq 1$, $s \in \mathbb{R}$, there holds the inequalities

$$
\begin{aligned}
&\langle (I + C_1)f, f\rangle_{L^2(\Gamma)} \geq \| f \|^2_{L^2(\Gamma)} - k_1(f,f) \\
(2.26)\qquad & \\
&\langle I + 2N_0)g, g\rangle_{L^2(\Gamma)} \geq \| g \|^2_{L^2(\Gamma)} - k_2(g,g),
\end{aligned}
$$

where $k_i(\cdot,\cdot)$ $(i = 1,2)$ is a compact bilinear form on $L^2(\Gamma) \times L^2(\Gamma)$.

The proof is obvious since the compactness of $k_1(\cdot,\cdot)$ is implied by the smoothness of C_1 and the compactness of $k_2(\cdot,\cdot)$ is implied by the smoothness of N_0.

3. The Galerkin Method for the Integral Equations

In this section we formulate the constructive solution of the system

$$
(3.1)\qquad \begin{pmatrix} I + C_1 & 0 \\ 0 & I + 2N_0 \end{pmatrix} \begin{pmatrix} f \\ g \end{pmatrix} = \begin{pmatrix} S_1(h_1,h_2) \\ S_2(h_1,h_2) \end{pmatrix} ,
$$

with Galerkin's procedure using one-dimensional regular finite element spaces $S_h^{t,k} := S_h^{t,k} \times S_h^{t,k} \subset H^k(\Gamma) \times H^k(\Gamma) = : \mathcal{H}^k$, $k \geq 0$, an integer, having the convergence property (3.3) and the inverse property (3.4) (see [1]). Due to a result by Hildebrandt and Wienholtz [4] the stability of the Galerkin operator (3.7) follows from Gårding's inequality (2.26). Thus, asymptotic error estimates with optimal order are valid for the Galerkin procedure corresponding to (3.1) (see [14]).

In the following we abbreviate (3.1) as

$$(3.2) \qquad BU = S, \quad \text{with } U = \begin{pmatrix} f \\ g \end{pmatrix} \in H^{s-1}(\Gamma) \times H^{s-1}(\Gamma) =: \mathcal{H}^{s-1}.$$

Then we assume for the components:

a) For any $v \in H^r(\Gamma)$ there exists a $\tilde{v} \in S_h^{t,k}$ with $t > r$ and a constant $C > 0$ independent of h and v such that for $q \leq \min\{k,r\}$

$$(3.3) \qquad \| v - \tilde{v} \|_{H^q(\Gamma)} \leq C h^{r-q} \| v \|_{H^r(\Gamma)} .$$

b) For $q \leq r$, $k \geq r$ there exists a constant $M > 0$ independent of h such that for all $\tilde{u} \in S_h^{t,k}$

$$(3.4) \qquad \| \tilde{u} \|_{H^r(\Gamma)} \leq M h^{q-r} \| \tilde{u} \|_{H^q(\Gamma)} .$$

Note that the parameter in $S_h^{t,k}$ have the following meaning:

i) h, $0 < h \leq 1$, is a parameter of mesh width,

ii) $t-1$ denotes the degree of the piecewise polynomials used as basic functions,

iii) k describes the conformity of the finite elements, i.e. $S_h^{t,k} \subset H^k(\Gamma)$ [1].

Together with the Gårding's inequality (2.26) this is sufficient to derive asymptotic error estimates in Sobolev norms for the Galerkin solution of the following problem:

Find $U_h := (f_h, g_h) \in \mathcal{S}_h^{t,k} \subset \mathcal{H}^k$ such that for all $\underset{\sim}{V} := (\underset{\sim}{\chi}, \underset{\sim}{\phi}) \in S_h^{t,k}$

$$(3.5) \qquad \langle BU_h, \underset{\sim}{V} \rangle_{\mathcal{L}^2(\Gamma)} = \langle BU, \underset{\sim}{V} \rangle_{\mathcal{L}^2(\Gamma)} = \langle S, \underset{\sim}{V} \rangle_{\mathcal{L}^2(\Gamma)} ,$$

where $\mathcal{L}^2(\Gamma) := L^2(\Gamma) \times L^2(\Gamma)$ and where $U = (f,g) \in \mathcal{H}^{s-1}$ is the exact solution of (3.2). More explicitly (3.5) reads

$$\text{(3.6)} \qquad \begin{aligned} \langle (I+C_1)f_h, \underset{\sim}{\chi}\rangle_{L^2(\Gamma)} &= \langle (I+C_1)f, \underset{\sim}{\chi}\rangle_{L^2(\Gamma)} = \langle S_1, \underset{\sim}{\chi}\rangle_{L^2(\Gamma)}, \\ \langle (I+2N_0)g_h, \underset{\sim}{\phi}\rangle_{L^2(\Gamma)} &= \langle (I+2N_0)g, \underset{\sim}{\phi}\rangle_{L^2(\Gamma)} = \langle S_2, \underset{\sim}{\phi}\rangle_{L^2(\Gamma)}, \end{aligned}$$

where $S = (S_1,S_2) \in \mathcal{H}^{s-1}$ is the given right hand side in (3.1).

<u>Theorem 3.1</u>: Let $t \geq 1$, $0 < h \leq 1$. Then the Galerkin operator

$$\text{(3.7)} \qquad G_B : U = (f,g) \to U_h = (f_h, g_h) : \mathcal{L}^2(\Gamma) \to \mathcal{L}^2(\Gamma),$$

defined by (3.5) is uniformly bounded independent of h. Moreover, we have the error estimates for $0 \leq \tau \leq r$

$$\text{(3.8)} \qquad \| U - U_h \|_{\mathcal{H}^\tau(\Gamma)} \leq C h^{r-\tau} \| U \|_{\mathcal{H}^r(\Gamma)},$$

where the constanct C is independent of h, U and U_h.

<u>Proof</u>: Due to Gårding's inequality (2.26) and the injectivity of the original system (3.1) a general result by Hildebrandt and Wienholtz [4] shows that the Galerkin solution U_h of (3.5) exists, is unique and converges to the exact solution U for $h \to 0$. Furthermore there exists a constant $C > 0$ independent of h such that

$$\text{(3.9)} \qquad \| G_B U \|_{\mathcal{L}^2(\Gamma)} \leq C \| U \|_{\mathcal{L}^2(\Gamma)}.$$

Using a technique by Nitsche [13] and the properties (3.3), (3.4) we obtain with (3.9) the desired estimate (3.8). This is shown by choosing an arbitrary element $\tilde{U} \in \mathcal{S}_h^{t,k}$ and using the projection property

$$G_B \tilde{U} = \tilde{U},$$

for any $\tilde{U} \in S_h^{t,k}$ of G_B. Thus

$$\begin{aligned}
\| U - U_h \|_{H^\tau(\Gamma)} &\leq \| U - G_B U + G_B \tilde{U} - \tilde{U} \|_{H^\tau(\Gamma)} \\
&\leq \| U - \tilde{U} \|_{H^\tau(\Gamma)} + \| G_B (U - U) \|_{H^\tau(\Gamma)} \\
&\leq C h^{r-\tau} \| U \|_{H^r(\Gamma)} + M h^{-\tau} \| G_B (\tilde{U} - U) \|_{L^2(\Gamma)} .
\end{aligned}$$

Now we use the uniformly boundedness (3.9) of the Galerkin operator G_B and can estimate further as follows.

$$\begin{aligned}
\| U - U_h \|_{H^\tau(\Gamma)} &\leq C h^{r-\tau} \| U \|_{H^r(\Gamma)} + \tilde{M} h^{-\tau} \| U - \tilde{U} \|_{L^2(\Gamma)} . \\
&\leq \tilde{C} h^{r-\tau} \| U \|_{H^r(\Gamma)} .
\end{aligned}$$

<u>Remark 3.2</u>: Obviously the estimate (3.8) holds also for the components itself due to the Gårding's inequalities (2.26) and the uncoupled form (3.1). So for example we have

$$\begin{aligned}
(3.10) \qquad \| f - f_h \|_{L^2(\Gamma)} &\leq C h^{s-1} \| f \|_{H^{s-1}(\Gamma)} \\
&\leq \tilde{C} h^{s-1} \{ \| h_1 \|_{H^s(\Gamma)} + \| h_2 \|_{H^{s-1}(\Gamma)} \},
\end{aligned}$$

due to the a priori estimate (2.24).

For $\Gamma \in C^{3+\varepsilon}$, $\varepsilon > 0$ there holds

$$h_1 := \psi_0 = \log|\underset{\sim}{x}(s) - \underset{\sim}{x}_0| \in C^{3+\varepsilon}(\Gamma), \quad h_2 := \frac{\partial\psi_0}{\partial n}\Big|_\Gamma \in C^{2+\varepsilon}(\Gamma).$$

Thus choosing $s-1 = 2+\varepsilon$ in (3.10) yields

$$\| f - f_h \|_{L^2(\Gamma)} \leq C\, h^{2+\varepsilon}$$

<u>Remark 3.3</u>: Thus we have constructed Galerkin approximations of the unknown densities f,g using the boundary element method. Hence, obviously approximations for the solution ψ of the interface problem (P_0) can be obtained by inserting the approximation of the densities into the single layer potential representations (1.2).

<u>Remark 3.4</u>: Inequality (3.10) clearly indicates the order of convergence depends on the smoothness of boundary data h_1, h_2. In our context of eddy current problems smoothness of the data is given by smoothness of the boundary. Thus the order of convergence of Galerkin's approximation is easily identified with the smoothness of the boundary.

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